

## Problem Set 4

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This fourth problem set explores set cardinality and graph theory. It serves as tour of the infinite (through set theory) and the finite (through graphs and their properties) and will give you a better sense for how discrete mathematical structures connect across these domains. Plus, you'll get to see some pretty pictures and learn about why all this matters in the first place. ☺

Some of the questions on this problem set will assume you've read the online Guide to Cantor's Theorem, which goes into more detail about the mechanics of the proof of Cantor's theorem as well as some auxiliary definitions.

Good luck, and have fun!

**Due Friday, May 3<sup>rd</sup> at 2:30PM.**  
**There is no checkpoint for pset4 (or any psets from now on).**

### Problem One: Cartesian Products and Set Cardinalities

If  $A$  and  $B$  are sets, the *Cartesian product* of  $A$  and  $B$ , denoted  $A \times B$ , is the set

$$\{ (x, y) \mid x \in A \wedge y \in B \}.$$

Intuitively,  $A \times B$  is the set of all ordered pairs you can make by taking one element from  $A$  and one element from  $B$ , in that order. For example, the set  $\{1, 2\} \times \{u, v, w\}$  is

$$\{ (1, u), (1, v), (1, w), (2, u), (2, v), (2, w) \}.$$

For the purposes of this problem, let's have  $\star$  and  $\odot$  denote two arbitrary objects where  $\star \neq \odot$ . Over the course of this problem, we're going to ask you to prove that  $|\mathbb{N} \times \{\star, \odot\}| = |\mathbb{N}|$ .

*For your own benefit, we strongly encourage you to pause and draw a picture showing a way to pair off the elements of  $\mathbb{N} \times \{\star, \odot\}$  with the elements of  $\mathbb{N}$  so that no elements of either set are uncovered or paired with multiple elements. You might want to draw some pictures of the set  $\mathbb{N} \times \{\star, \odot\}$  so that you can get a better visual intuition. You do not need to turn it in.*

- i. Based on the picture you came up with, define a bijection  $f: \mathbb{N} \times \{\star, \odot\} \rightarrow \mathbb{N}$ . The inputs to this function will be elements of  $\mathbb{N} \times \{\star, \odot\}$ , so you can define your function by writing

$$f(n, x) = \quad [fill\ in\ blank\ here]$$

where  $n \in \mathbb{N}$  and  $x \in \{\star, \odot\}$ .

*In defining this function, you cannot assume  $\star$  or  $\odot$  are numbers, since they're arbitrary values out of your control. See if you can find a way to define this function that doesn't treat  $\star$  and  $\odot$  algebraically.*

- ii. Prove that the function you came up with in part (i) is a bijection.

The result you've proved here essentially shows that  $2\aleph_0 = \aleph_0$ . Isn't infinity weird?

## Problem Two: Understanding Diagonalization

Proofs by diagonalization are tricky and rely on nuanced arguments. In this problem, we'll ask you to review the formal proof of Cantor's theorem to help you better understand how it works.

(Please read the Guide to Cantor's Theorem before attempting this problem.)

- i. Consider the function  $f: \mathbb{N} \rightarrow \wp(\mathbb{N})$  defined as  $f(n) = \emptyset$ . *This is of course a **terrible** proposal for  $f$ , if the point is to come up with a bijection between  $\mathbb{N}$  and  $\wp(\mathbb{N})$ . It's not even close to injective (every single value of  $n$  from the domain  $\mathbb{N}$  gives the same  $f(n)$ ), and it's not even close to surjective either (only one element of the co-domain  $\wp(\mathbb{N})$  is used). But we can't really blame the person who came up with it—we know that there **is no** such bijection, so the only surprising thing about this not being one is just how spectacularly it fails.* Despite how poor this  $f$  is, tracing through our formal proof of Cantor's theorem with this choice of  $f$  in mind will help us see with a concrete example  $f$  how the set  $D$  is made. So go ahead and trace through our formal Cantor proof, to the part where the proof defines some set  $D$  in terms of  $f$ . Given that  $f(n) = \emptyset$ , what is that set  $D$  and how do we know that  $f(n) \neq D$  for any  $n \in \mathbb{N}$ ? Write  $D$  without using set-builder notation (there is an extremely concise solution), and you just need one brief sentence to explain how we know that  $f(n) \neq D$  for any  $n \in \mathbb{N}$ .

*Make sure you can determine what the set  $D$  is both by using the visual intuition behind Cantor's theorem and by symbolically manipulating the formal definition of  $D$  given in the proof.*

- ii. Let  $f$  be the function from part (i). Find a set  $S \subseteq \mathbb{N}$  such that  $S \neq D$ , but  $f(n) \neq S$  for any  $n \in \mathbb{N}$ . Justify your answer. This shows that while the diagonalization proof will always find *some* set  $D$  that isn't covered by  $f$ , it won't find *every* set with this property.
- iii. Repeat part (i) of this problem using the function  $f: \mathbb{N} \rightarrow \wp(\mathbb{N})$  defined as
 
$$f(n) = \{ m \in \mathbb{N} \mid m \geq n \}$$
 Now what do you get for the set  $D$ ? (ok to use set-builder this time). Briefly describe in words why  $f(n) \neq D$  for any  $n \in \mathbb{N}$ .
- iv. Repeat part (ii) of this problem using the function  $f$  from part (iii).
- v. Give a function  $f: \mathbb{N} \rightarrow \wp(\mathbb{N})$  such that the set  $D$  obtained from the proof of Cantor's theorem is the set  $\{ n \in \mathbb{N} \mid n \text{ is even} \}$ . Briefly justify your answer.

## Problem Three: Simplifying Cantor's Theorem?

Below is a purported proof that  $|S| \neq |\wp(S)|$  that doesn't use a diagonal argument:

**Theorem:** If  $S$  is a set, then  $|S| \neq |\wp(S)|$ .

**Proof:** Let  $S$  be any set and consider the function  $f: S \rightarrow \wp(S)$  defined as  $f(x) = \{x\}$ . To see that this is a valid function from  $S$  to  $\wp(S)$ , note that for any  $x \in S$ , we have  $\{x\} \subseteq S$ . Therefore,  $\{x\} \in \wp(S)$  for any  $x \in S$ , so  $f$  is a legal function from  $S$  to  $\wp(S)$ .

Let's now prove that  $f$  is injective. Consider any  $x_1, x_2 \in S$  where  $f(x_1) = f(x_2)$ . We'll prove that  $x_1 = x_2$ . Because  $f(x_1) = f(x_2)$ , we have  $\{x_1\} = \{x_2\}$ . Since two sets are equal if and only if their elements are the same, this means that  $x_1 = x_2$ , as required.

However,  $f$  is not surjective. Notice that  $\emptyset \in \wp(S)$ , since  $\emptyset \subseteq S$  for any set  $S$ , but that there is no  $x$  such that  $f(x) = \emptyset$ ; this is because  $\emptyset$  contains no elements and  $f(x)$  always contains one element. Since  $f$  is not surjective, it is not a bijection. Thus  $|S| \neq |\wp(S)|$ . ■

Unfortunately, this argument is incorrect. What's wrong with this proof? Justify your answer by pointing to a specific incorrect claim that's made here and explaining why it's incorrect.

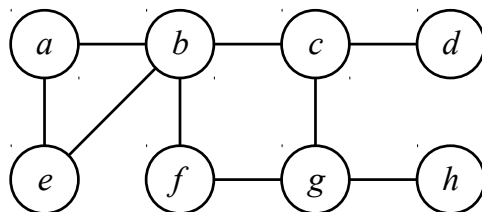
## Problem Four: Independent and Dominating Sets

An **independent set** in a graph  $G = (V, E)$  is a set  $I \subseteq V$  with the following property:

$$\forall u \in I. \forall v \in I. \{u, v\} \notin E.$$

Let's begin with a quick warm-up about independent sets.

- i. Consider the graph shown below. Give two different independent sets of this graph, each of which has cardinality three or greater. No justification is necessary.



Now, a new definition. A **dominating set** in  $G$  is a set  $D \subseteq V$  with the following property:

$$\forall v \in V. (v \notin D \rightarrow \exists u \in D. \{u, v\} \in E)$$

As above, it's good to play around with this definition a bit before moving on.

- ii. Give two different examples of dominating sets of the above graph, each of which has cardinality four or less. No justification is necessary.
- iii. Let  $G = (V, E)$  be a graph with the following property: every node in  $G$  is adjacent to at least one other node in  $G$ . Prove that if  $I$  is an independent set in  $G$ , then  $V - I$  is a dominating set in  $G$ .

*Notice that we're asking you to show that  $V - I$  is a dominating set, not that  $I$  is a dominating set. Also, we recommend drawing some pictures here to get a sense of how this works. After all, you have a couple of examples of independent sets from part (i) of this problem!*

*Use the formal definitions to guide your proofs. If you proceed via a direct proof or via contrapositive, what, exactly, will you be assuming, and what will you be proving? If you write this as a proof by contradiction, what specifically is it that you're assuming for the sake of contradiction?*

An independent set  $I$  in a graph  $G$  is a **maximal independent set** in  $G$  if there is no independent set  $I'$  in  $G$  where  $I \subsetneq I'$ . (Here,  $I \subsetneq I'$  denotes that  $I$  is a strict subset of  $I'$ ).

- iv. Find independent sets  $I$  and  $J$  of the graph from part (i) of this problem such that  $I$  is maximal but  $|I| < |J|$ . No justification is necessary.

*Yes, this is possible. The definition of a maximal independent set is meant to be taken literally.*

- v. Prove that if  $I$  is a maximal independent set in  $G = (V, E)$ , then  $I$  is a dominating set of  $G$ .

You can build a great intuition for this result by drawing some pictures and thinking about what has to happen for a set of nodes to be an independent set and for a set of nodes to be a dominating set. When it comes time to write out your proof, however, you'll need to use the formal first-order definitions of independent sets, maximal independent sets, and dominating sets.

## Problem Five: Highly Irregular Graphs

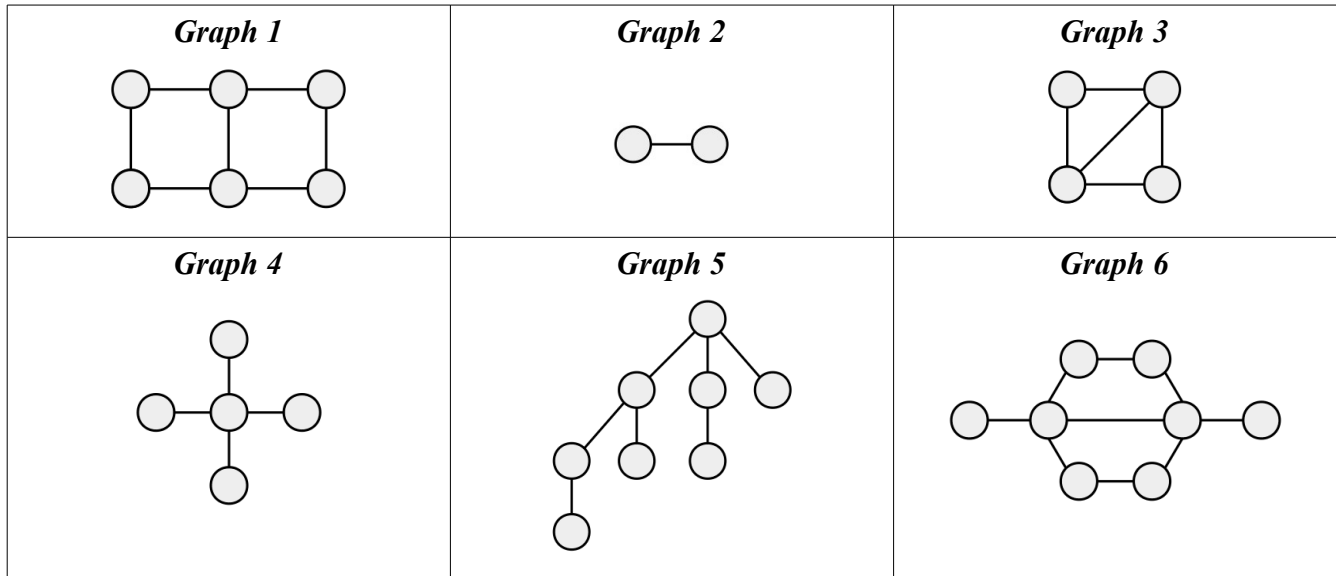
As a refresher, the *degree* of a node in a graph  $G$ , denoted  $\text{deg}(v)$ , is the number of nodes that  $v$  is adjacent to. Equivalently, it's the number of edges touching  $v$ .

Now, a new definition. A graph  $G = (V, E)$  is called *highly irregular* if this first-order formula is true:

$$\forall v \in V. \forall x \in V. \forall y \in V. (x \neq y \wedge \{v, x\} \in E \wedge \{v, y\} \in E \rightarrow \text{deg}(x) \neq \text{deg}(y)).$$

That definition might look like a mouthful, but it's actually not that bad once you get the hang of it.

- i. Below is a collection of graphs. Which ones are highly irregular? No justification is necessary.



*Since the definition of highly irregular graphs depends on the degrees of the nodes, it's probably not a bad idea to annotate each node with its degree.*

Highly irregular graphs have some interesting properties. First, a new definition. If  $G$  is a graph with at least one node, then we'll have  $\Delta(G)$  denote the maximum degree of any of the nodes in  $G$ .

- ii. Draw the simplest possible graph  $G$  that has at least two nodes but only one node of degree  $\Delta(G)$ . By "simplest," we mean having as few nodes as possible, then as few edges as possible among all graphs with the smallest number of nodes. No justification is necessary.

Although graphs in general can have exactly one degree of node  $\Delta(G)$ , highly irregular graphs cannot.

- iii. Prove that if  $G$  is a highly irregular graph and  $\Delta(G) \geq 2$ , then  $G$  has at least two nodes of degree  $\Delta(G)$ . (This theorem is also true in the case where  $\Delta(G) = 0$  or  $\Delta(G) = 1$ , but these are somewhat degenerate cases and aren't as interesting.)

*As a hint, proceed by contradiction. Look back at the pictures above that you identified as highly irregular graphs – do you notice anything about where the nodes of degree  $\Delta(G)$  are? See if you can use that to build out your proof.*

As a refresher, a *triangle* in a graph is a group of three different nodes  $x, y$ , and  $z$  where  $\{x, y\}$ ,  $\{y, z\}$ , and  $\{x, z\}$  are all edges in the graph.

- iv. Draw a highly irregular graph that contains a triangle. To make things easier for the TAs to grade, please annotate each node with its degree and highlight three nodes that form a triangle. For full credit, your solution should have ten nodes or fewer. No justification is necessary.

*Something to think about: if  $G$  is highly irregular and  $G$  contains a triangle, what is the smallest possible value for  $\Delta(G)$ ? Based on that and your observations from part (iii) of this problem, what does that tell you about the shape of the graph? Use that to guide your search.*

## Problem Six: Bipartite Graphs

There are a few famous families of graphs that come up over and over again. One of the most important types of graphs in computer science is the *bipartite graph*, which is the focus of this problem.

Let's begin with a formal definition of bipartite graphs. An undirected graph  $G = (V, E)$  is called *bipartite* if there exist two sets  $V_1$  and  $V_2$  such that

- every node  $v \in V$  belongs to exactly one of  $V_1$  and  $V_2$ , and
- every edge  $e \in E$  has one endpoint in  $V_1$  and the other in  $V_2$ .

The sets  $V_1$  and  $V_2$  here are called *bipartite classes* of  $G$ . To help you get a better sense for why bipartite graphs are important and where they show up, let's work through a couple of examples.

- i. Consider a graph where each node represents a square on a chessboard and where there's an edge between any pair of squares that are immediately adjacent in one of the four cardinal directions (up, down, left, and right). Explain why this is a bipartite graph by telling us what the bipartite classes are and briefly explaining why all the edges have one endpoint in each bipartite class.

*Suggestion: draw lots of pictures!*

Bipartite graphs have many interesting properties. One of the most fundamental is this one:

**Theorem:** An undirected graph  $G$  is bipartite if and only if it contains no cycles of odd length.

The forward direction of this implication has a nice intuition.

*Pause to see if you can convince yourself, intuitively, why if  $G$  is bipartite, then it has no cycles of odd length. Specifically, why every cycle in  $G$  has to have even length.*

The reverse direction of this implication – that if  $G$  has no cycles of odd length, then  $G$  is bipartite – requires a different sort of argument.

Let's pick some arbitrary graph  $G = (V, E)$  that has no cycles of odd length. For simplicity's sake, we'll assume that  $G$  has just one connected component. If  $G$  has two or more connected components, we can essentially treat each one of them as independent graphs. (*Do you see why?*)

Now, choose any node  $v \in V$ . Using node  $v$  as an "anchor point," we can define two sets  $V_1$  and  $V_2$  that we'll need for the remainder of this argument:

$$V_1 = \{ x \in V \mid \text{there is an odd-length path from } v \text{ to } x \}$$

$$V_2 = \{ x \in V \mid \text{there is an even-length path from } v \text{ to } x \}$$

This turns out to be a really useful way to group the nodes of  $G$ .

- ii. Given the choices of  $G$  and  $v$  from above, prove that  $V_1$  and  $V_2$  have no nodes in common.

*Remember that there might be multiple different paths of different lengths from  $v$  to some other node  $x$ , so be careful not to talk about "the" path between  $v$  and  $x$ . Also note that these **do not** have to be simple paths.*

- iii. Using your result from part (ii), prove that  $G$  is bipartite.

*The most common mistake on this problem is to not address all the parts of the definition of a bipartite graph. So start off by writing down a list of what you need to prove, then address each part in turn.*

On this problem set, we've included three optional fun problems you can play around with. You're welcome to play around with any number of them, but ***please submit answers to at most one of these problems*** with your problem set. We, unfortunately, don't have TA bandwidth to grade both of them. If you submit solutions to more than one of them, we'll choose one to grade arbitrarily.

### Optional Fun Problem One: Hugs All Around! (Extra Credit)

There's a party with 137 attendees. Each person is either *honest*, meaning that they *always* tell the truth, or *mischievous*, meaning that they *never* tell the truth. After everything winds down, everyone is asked how many *honest* people they hugged at the party. Surprisingly, each of the numbers 0, 1, 2, 3, ... and 136 was given as an answer exactly once.

How many honest people were at the party? Prove that your answer is correct and that no other answer could be correct.

### Optional Fun Problem Two: How Many Functions Are There? (Extra Credit)

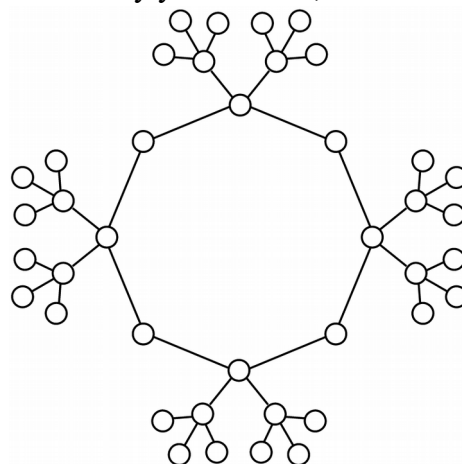
If  $A$  and  $B$  are sets, we can define the set  $B^A$  to be the set of all functions from  $A$  to  $B$ . Formally speaking:

$$B^A = \{ f \mid f: A \rightarrow B \}$$

Prove that  $|\mathbb{N}| < |\mathbb{N}^{\mathbb{N}}|$ . This shows that  $\aleph_0 < \aleph_0^{\aleph_0}$ . Isn't infinity weird?

### Optional Fun Problem Three: Chemical Automorphisms

Below is a graph of the molecular structure of octomethylcyclotetrasiloxane. How many automorphisms does this graph have? Justify your answer, but no formal proof is required.



### Optional Fun Problem Four: Chromatic and Independence Numbers

Recall that if  $G$  is a graph, then  $\chi(G)$  represents the *chromatic number* of  $G$ , the minimum number of colors needed to paint each node of  $G$  so no two adjacent nodes of  $G$  are the same color. The *independence number* of a graph, denoted  $\alpha(G)$ , is the size of the largest independent set in  $G$ .

Let  $n$  be an arbitrary positive natural number. Prove that if  $G$  is an arbitrary undirected graph with exactly  $n^2+1$  nodes, then  $\chi(G) \geq n+1$  or  $\alpha(G) \geq n+1$  (or both).

*You should definitely check out the [Guide to Proofs on Discrete Structures](#)' advice for proving  $P \vee Q$ .*